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CAVITIES *VIS-A-VIS* RIGID INCLUSIONS: ELASTIC MODULI OF MATERIALS WITH POLYGONAL INCLUSIONS

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Abstract—In this paper we explore the correspondence between rigid inclusions and cavities [Dundurs (1989, *J. Appl. Mech.* **56**, 786–790)] as applied to the effective elastic moduli of materials with polygonal rigid inclusions and cavities. In the analysis we use a complex variable method of elasticity and a conformal transformation to solve for the stress field due to a single rigid inclusion. Then we use a *far field* approach to obtain the effective elastic constants of composites with a dilute concentration of rigid polygonal inclusions. By employing the Dundurs correspondence we can automatically obtain the result for the effective elastic moduli of materials with cavities. Finally, we use effective medium theories to predict the elastic moduli of materials containing a finite concentration of inclusions.

1. INTRODUCTION

We study effective elastic moduli of materials containing rigid polygonal inclusions and cavities (holes). First, we find stress and displacement fields due to a single polygonal rigid inclusion using a complex variable method of elasticity (Muskhelishvili, 1953; Savin, 1961) and a conformal transformation. Then we obtain the effective elastic constants of materials with a dilute concentration of inclusions by using a far field approach. From these results we can find the effective elastic moduli of materials with polygonal holes by using the Dundurs correspondence (Dundurs, 1989) between the solutions of stress fields for rigid inclusions and cavities. Finally, we use the dilute result in effective medium theories and predict the effective elastic moduli of materials having a finite concentration of either rigid inclusions or holes.

This paper extends our earlier studies in which we obtained effective elastic moduli of materials with polygonal holes (Jasiuk *et al.*, 1992b, 1994) and rigid circular inclusions (Jasiuk *et al.*, 1992a). Other related works are due to Kachanov (1993) and Kachanov *et al.* (1994), who studied the effective elastic constants of materials with cavities and cracks, Zimmerman (1986), who investigated the plane elasticity problem of compressibility of materials with pores of arbitrary shapes, and Movchan and Serkov (1992) and Movchan (1992) who used the concept of an elastic polarization matrix to capture far field effects of inclusions in a material. Several solutions for polygonal rigid inclusions and holes are included in Savin (1961) and a problem of a rigid square inclusion was studied in detail by Chang and Conway (1968).

2. THE PLANE ELASTICITY

The strain–stress relations for a linear elastic and isotropic material for the plane elasticity are given by

Table 1. Relations between elastic constants

Elastic constants	General relations	Plane stress	Plane strain	Other relations
Kolosov constant η	η	$\frac{3-\nu}{1+\nu}$	$3-4\nu$	$\eta = 1+\frac{2G}{K}$
Uniaxial elastic modulus E	$\frac{8G}{\eta+1}$	E'	$\frac{E'}{1-\nu^2}$	$\frac{4}{\frac{1}{K}+\frac{1}{G}}$
Area bulk modulus K	$\frac{2G}{\eta-1}$	$\frac{E'}{2(1-\nu)}$	$\frac{E'}{2(1+\nu)(1-2\nu)}$	
Shear modulus G	G	$\frac{E'}{2(1+\nu)}$	$\frac{E'}{2(1+\nu)}$	

$$\varepsilon_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{3-\eta}{4}\sigma_{kk}\delta_{ij}\right], \quad i,j,k = 1,2, \tag{1}$$

where ε_{ij} and σ_{ij} are the strain and stress tensors, respectively, G is the shear modulus and η is a Kolosov constant defined as

$$\eta = 3-4\nu \quad \text{(plane strain)}$$
$$\eta = \frac{3-\nu}{1+\nu} \quad \text{(plane stress)}, \tag{2}$$

where ν is the Poisson's ratio and $1 \leqslant \eta \leqslant 3$ for plane strain and $5/2 \leqslant \eta \leqslant 3$ for plane stress, if we restrict ourselves to positive values of Poisson's ratio. In a purely two-dimensional case, when a two-dimensional Poisson's ratio, having an upper bound of unity, is used (Thorpe and Jasiuk, 1992), the limits are $1 \leqslant \eta \leqslant 3$.

It is convenient to introduce planar elastic constants: a uniaxial elastic modulus E and an area bulk modulus K. Following Dundurs and Markenscoff (1993) and Jasiuk *et al.* (1994) they are defined as

$$E = \frac{8G}{\eta+1} \tag{3}$$

$$K = \frac{2G}{\eta-1}. \tag{4}$$

Two other useful relations are

$$\frac{4}{E} = \frac{1}{K}+\frac{1}{G} \tag{5}$$

$$\eta = 1+2\frac{G}{K}. \tag{6}$$

For convenience, we include a more complete summary of relations between the elastic constants in Table 1, where we denote the Young's modulus by E' to avoid conflict of notation with the above defined planar elastic modulus E.

3. THE SINGLE INCLUSION SOLUTION

We first solve a plane elasticity problem of a single rigid polygonal inclusion embedded in an isotropic, homogeneous and linear elastic material subjected to a uniform remote uniaxial tension. In the analysis we use the complex variable method of elasticity involving the Schwarz–Christoffel conformal mapping (Muskhelishvili, 1953; Savin, 1961; Sokolnikov, 1986).

In the complex variable theory, if we have two complex domains S and Σ in the z and ζ planes, respectively, the conformal transformation is given by

$$z = w(\zeta), \tag{7}$$

where $\zeta = \rho\, e^{i\upsilon}$ and ρ and υ are polar coordinates. If the domains S and Σ are infinite, transformation (7) is of the form

$$w(\zeta) = R\left(\zeta + \sum_{n=1}^{\infty} a_n \zeta^{-n}\right) \tag{8}$$

and if the exterior of a regular polygon in the z plane is transformed into the exterior of a unit circle in the ζ plane, eqn (8) becomes

$$w(\zeta) = R\left(\zeta + \frac{2}{n(n-1)\zeta^{n-1}} + \frac{n-2}{n^2(2n-1)\zeta^{2n-1}} + \frac{(n-2)(2n-2)}{3n^3(3n-1)\zeta^{3n-1}} + \cdots\right), \tag{9}$$

where $n = 3, 4, 5, \ldots$ (n is the number of sides in a polygon) and R is a constant. Theoretically, the polygon requires an infinite number of terms in the series (9); otherwise the corners are rounded. However, the shape of the polygon is approximated well with just a few terms (Jasiuk *et al.*, 1994). In the numerical examples presented in this paper we include at most the first three nontrivial terms as shown in eqn (9). We will demonstrate in Section 4 that using only the first nontrivial term in the series will give a very good estimate of the effective elastic moduli of a material with polygonal inclusions, and the addition of extra terms will give only a small correction.

The complex variable method in elasticity involves the determination of two analytic functions $\phi(\zeta)$ and $\psi(\zeta)$, which represent stress functions. The stresses and displacements, in polar coordinates, are expressed in terms of these functions as

$$\sigma_{\rho\rho} + \sigma_{\upsilon\upsilon} = 4 Re\, \Phi(\zeta) \tag{10}$$

$$\sigma_{\upsilon\upsilon} - \sigma_{\rho\rho} + 2i\sigma_{\rho\upsilon} = \frac{2\zeta^2}{\rho^2 \overline{w'(\zeta)}}[\overline{w(\zeta)}\Phi'(\zeta) + w'(\zeta)\Psi(\zeta)] \tag{11}$$

$$2G|w'(\zeta)|(u_\rho + iu_\upsilon) = \frac{\bar{\zeta}}{\rho}\overline{w'(\zeta)}\left[\eta\phi(\zeta) - \frac{w(\zeta)}{\overline{w'(\zeta)}}\overline{\phi'(\zeta)} - \overline{\psi(\zeta)}\right], \tag{12}$$

where bars denote a conjugate and $\Phi(\zeta)$ and $\Psi(\zeta)$ are given by

$$\Phi(\zeta) = \frac{\phi'(\zeta)}{w'(\zeta)}, \qquad \Psi(\zeta) = \frac{\psi'(\zeta)}{w'(\zeta)}. \tag{13}$$

The boundary conditions on the surface Γ of the rigid inclusion, given by $\sigma = e^{i\upsilon}$ where $\rho = 1$, involve the displacements and are of the form

$$\eta\phi(\sigma)-\frac{w(\sigma)}{\overline{w'(\sigma)}}\overline{\phi'(\sigma)}-\overline{\psi(\sigma)}=2Gg(\sigma), \tag{14}$$

where

$$g(\sigma)=u_x+iu_y \tag{15}$$

and u_x and u_y are the Cartesian components of the displacement vector acting on the boundary Γ. In the problems considered here, these displacements are zero except for the case of a rigid triangle which involves a rigid body displacement translation.

It is convenient to write the stress functions $\phi(\zeta)$ and $\psi(\zeta)$ in the form

$$\phi(\zeta)=A_0R\zeta+\phi_1(\zeta) \tag{16}$$

$$\psi(\zeta)=B_0R\zeta+\psi_1(\zeta), \tag{17}$$

where

$$\phi_1(\zeta)=\sum_{n=1}^{\infty}d_n\zeta^{-n},\qquad \psi_1(\zeta)=\sum_{n=1}^{\infty}e_n\zeta^{-n} \tag{18}$$

and for a uniaxial tension case with $\sigma^0_{xx}=T$

$$A_0=\frac{T}{4},\qquad B_0=\frac{-T}{2}. \tag{19}$$

If we substitute eqns (16) and (17) into the boundary condition (14) we have

$$\eta\phi_1(\sigma)-\frac{w(\sigma)}{\overline{w'(\sigma)}}\overline{\phi_1'(\sigma)}-\overline{\psi_1(\sigma)}=2Gg_1(\sigma), \tag{20}$$

where

$$2Gg_1(\sigma)=TR\left[-\frac{\eta\sigma}{4}+\frac{1}{4}\frac{w(\sigma)}{\overline{w'(\sigma)}}-\frac{1}{2\sigma}\right]. \tag{21}$$

If we multiply both sides of eqn (20) and its conjugate equation by $1/[2\pi i(\sigma-\zeta)]$ and integrate over the contour Γ, we obtain two equations for the two unknown stress functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$:

$$\phi_1(\zeta)=\frac{1}{\eta}\left[-\frac{1}{2\pi i}\oint_\Gamma\frac{g_1(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma-\frac{1}{2\pi i}\oint_\Gamma\frac{w(\sigma)}{\overline{w'(\sigma)}}\frac{\overline{\phi_1'(\sigma)}}{\sigma-\zeta}\mathrm{d}\sigma\right] \tag{22}$$

$$\psi_1(\zeta)=\frac{1}{2\pi i}\oint_\Gamma\frac{\overline{g_1(\sigma)}}{\sigma-\zeta}\mathrm{d}\sigma+\frac{1}{2\pi i}\oint_\Gamma\frac{\overline{w(\sigma)}}{w'(\sigma)}\frac{\phi_1'(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma. \tag{23}$$

After the stress functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ are determined, the stresses and displacements are found using eqns (10)–(12). We have conducted these calculations using the symbolic algebra program MAPLE (1992). We express these stresses and displacements, for convenience in the later analysis, in terms of powers of $1/r$; they are in the form of infinite series.

When a circular rigid inclusion or a hole is embedded in an infinite material and subjected to a remote uniaxial tension $\sigma_{xx}^0 = T$, the solution can be obtained by using the Airy stress function [see, for example, Dundurs (1989)]

$$U = \frac{T}{4}\left(r^2 - r^2 \cos 2\theta - 2c \log r + 2d \cos 2\theta - e\frac{\cos 2\theta}{r^2}\right), \tag{24}$$

where c, d and e are constants. Following Jasiuk *et al.* (1994), when the inclusion is in the shape of a regular polygon the stresses and displacements involve an infinite number of terms in the series, as opposed to a finite number of terms for a circular inclusion, but the leading terms are of the same form as those obtained from eqn (24). They are given as

$$\sigma_{rr} = \frac{T}{2}\left[1 + \cos 2\theta - \frac{c + 4d \cos 2\theta}{r^2} + \cdots\right] \tag{25}$$

$$\sigma_{r\theta} = -\frac{T}{2}\left[\sin 2\theta + \frac{2d \sin 2\theta}{r^2} + \cdots\right] \tag{26}$$

$$\sigma_{\theta\theta} = \frac{T}{2}\left[1 - \cos 2\theta + \frac{c}{r^2} + \cdots\right] \tag{27}$$

$$2Gu_r = \frac{TR}{2}\left[\frac{(\eta - 1)}{2} r + r \cos 2\theta + \frac{c}{r} + \frac{d(\eta + 1) \cos 2\theta}{r} + \cdots\right] \tag{28}$$

$$2Gu_\theta = \frac{-TR}{2}\left[r \sin 2\theta + d(\eta - 1)\frac{\sin 2\theta}{r} + \cdots\right]. \tag{29}$$

These results for the far field [eqns (25)–(29)] determine completely the effective elastic moduli of composite material with polygonal inclusions as shown in Section 4. It should be pointed out that this far field approach involves no approximations and is exact.

4. THE DILUTE RESULT FOR EFFECTIVE ELASTIC MODULI

When the material has a very small concentration of inclusions and there is no interaction between the inclusions, the effective elastic moduli of such a material can be calculated exactly. This dilute result can be found by using a single inclusion solution obtained in section 3 and, for example, the equivalence of elastic strain energies (Christensen, 1979).

When a remote stress field σ_{ij}^0 is applied to the domain V having a single inclusion Ω, then the elastic strain energy W^c of this composite material is expressed as

$$W^c = W^0 + \frac{1}{2}\int_{|\Omega|} (\sigma_{ij}^0 n_j u_i - \sigma_{ij} n_j u_i^0)\, \mathrm{d}S, \tag{30}$$

where $W^0 = \frac{1}{2}\int_V \sigma_{ij}^0 \varepsilon_{ij}^0 \, \mathrm{d}V$ and $|\Omega|$ is the boundary of the inclusion. The superscript zero specifies the undisturbed quantities due to the applied load, while the quantities σ_{ij} and u_i denote the total stresses and displacements which include both σ_{ij}^0, u_i^0 and the local disturbance caused by the inclusion. n_j is the unit vector normal to the inclusion–matrix interface.

The elastic strain energy stored in the equivalent homogeneous medium, having the properties of a composite, is given as

$$W^c = \tfrac{1}{2}\sigma_{ij}^0 S_{ijkl}^c \sigma_{kl}^0 V, \tag{31}$$

where S_{ijkl}^c is the effective compliance and V is the volume (area) of the composite material. Then, the effective elastic properties of the material with inclusions can be found by equating the elastic strain energies of heterogeneous and homogeneous systems, given by eqns (30) and (31), respectively. We refer to this method as the near field approach.

In our analysis we use the formula (30) but take the integral over a very *large surface* $|\Omega_0|$, which is circular for convenience and has a radius r_0:

$$W^c = W^0 + \frac{1}{2}\int_0^{2\pi} [\sigma_{rr}^0 u_r + \sigma_{r\theta}^0 u_\theta - (\sigma_{rr} u_r^0 + \sigma_{r\theta} u_\theta^0)]|_{r=r_0} r_0 \, d\theta. \tag{32}$$

If we substitute stresses and displacements [eqns (25)–(29)] obtained in section 3 into eqn (32), only the stress terms with $1/r^2$ and displacements with $1/r$ will contribute towards the effective elastic moduli. This is true for the inclusions of both circular and polygonal shapes. This far field approach has been used by Thorpe (1992) for dielectric problems and Jasiuk *et al.* (1994) for elastic materials with polygonal holes. The near and far field methods yield the same results for the effective elastic moduli but we find the far field approach to be simpler.

When we evaluate the effective planar uniaxial elastic modulus E^c, for example, of a material with polygonal inclusions, we find

$$\frac{E}{E^c} = \left[1 + (c+2d)\frac{A_{\text{circle}}}{A_{\text{polygon}}} f\right], \tag{33}$$

where A denotes area, f is the area fraction of polygons defined as $f = A_{\text{polygon}}/A_{\text{total}}$, E is the uniaxial elastic modulus of the matrix material and c and d are constants which were introduced in section 3. These constants depend on η for the rigid inclusions case and are independent of η for the case of cavities. Equation (33) reduces to the expression for the rigid circular inclusions when $c = (1-\eta)/2$, and $d = -(1/\eta)$, and when A_{polygon} is replaced by A_{circle}.

The ratio of the area of a circle to that of a polygon is given by

$$\frac{A_{\text{circle}}}{A_{\text{polygon}}} = \frac{1}{1 - \sum_{m=1}^{\infty} m|a_m|^2}, \tag{34}$$

where a_m is defined in eqn (9) and

$$1 - \sum_{m=1}^{\infty} m|a_m|^2 = \left[1 - \frac{4}{n^2(n-1)} - \frac{(n-2)^2}{n^4(2n-1)} - \frac{(n-2)^2(2n-2)^2}{9n^6(3n-1)} - \cdots\right], \tag{35}$$

where n denotes the number of sides in a polygon.

For simplicity of notation we introduce two constants $\tilde{c}$ and $\tilde{d}$ defined as

$$\tilde{c} = c\frac{A_{\text{circle}}}{A_{\text{polygon}}}, \quad \tilde{d} = d\frac{A_{\text{circle}}}{A_{\text{polygon}}}. \tag{36}$$

These constants are given in the Appendix for several polygonal shapes.

In Jasiuk *et al.* (1992b, 1994) we have introduced the parameter α defined by

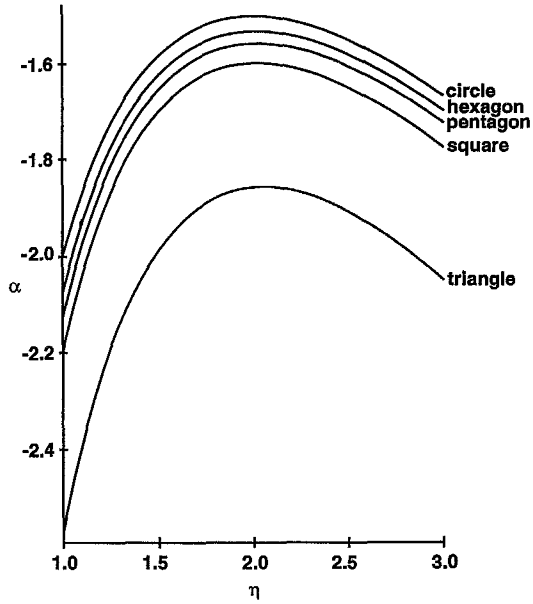

Fig. 1. The parameter α versus η for materials with rigid inclusions of several polygonal shapes approximated by one nontrivial term in eqn (9).

$$\alpha = \tilde{c}+2\tilde{d}. \tag{37}$$

Then the dilute uniaxial elastic modulus can be written as

$$\frac{E}{E^{c}} = 1+\alpha f. \tag{38}$$

The parameter α is of interest to us since for materials with holes it is independent of the elastic constants and thus the effective uniaxial modulus E^{c}, given in eqn (38), is independent of the Poisson's ratio of the matrix and only depends on the shape of cavities. This result is rigorous and holds for an arbitrary geometry and area fraction as proved by Cherkaev *et al.* (1992) and Day *et al.* (1992) and discussed by Thorpe and Jasiuk (1992). In the case of rigid inclusions the parameter α is a function of Poisson's ratio. The numerical results for α are given in Figs 1 and 2.

Figure 1 illustrates how α varies as a function of η for several polygonal shapes: the equilateral triangle, square, pentagon, hexagon, approximated by the transformation (9) involving only the first nontrivial term, and the circle (being a limit case of an n-sided polygon). It is interesting to note that for the rigid inclusion shapes considered here the material with a rigid triangle yields higher modulus E^{c} than the other shapes and circle gives least reinforcement for the same area fraction. This is due to the fact that the stiff and pointed corners of the triangle constrain material deformation more than the more circular-like shapes and thus effectively the material reinforced with rigid triangles is stiffer than the one containing other polygonal shapes discussed here. This behavior is opposite in the case of cavities, where the triangular shape gives the lowest effective modulus E^{c} (Jasiuk *et al.*, 1992b, 1994; Kachanov *et al.*, 1994). In this situation the corners of the triangle have a softening effect, they allow more deformation and they make the material more flexible than the other polygonal shapes do. A similar observation has been made by Kachanov *et al.* (1994) who also discussed the influence of geometrical details of holes on the effective moduli. They compared the polygons with smooth corners with the corresponding hypotrochoids of the same area having sharp corners and observed that the shapes with sharp corners gave a much higher compliance than those with the smooth corners.

Another observation about rigid inclusions is that there is a large dependence of α on the Poisson's ratio and that there is an optimum value of η at which α is minimum and thus

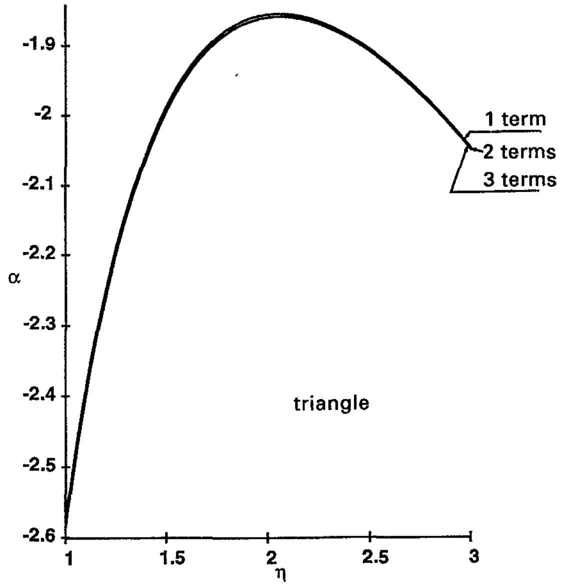

Fig. 2. The parameter α versus η for materials with triangular rigid inclusions approximated by one, two and three nontrivial terms in eqn (9).

the modulus E^c is the highest. This value is at $\eta = 1$, which corresponds to $v = 1/2$ for the plane strain case. Also, α may have the same value for two different values of Poisson's ratio.

We should comment that a square is not an isotropic shape and in Fig. 1 we obtain the parameter α for the square by averaging α over two positions differing by 45° [more details on this procedure are given in Jasiuk *et al.* (1994)].

Figure 2 illustrates the effect of η on α for the case of a triangle which was approximated by one, two and three nontrivial terms in the conformal transformation (9). The increased number of terms in the transformation implies that the edges of the triangle are more straight and the corners are sharper. This sharpening of the corners has a significant influence on the local stress fields; the smaller the curvature at corners, the higher is the stress in the vicinity of corners, as explored in Chang and Conway (1968) and Savin (1961). Thus, the plasticity or cracking will first initiate there. However, this effect of sharp corners is local and has a very small influence on the overall elastic moduli as shown in Fig. 2. For example, for the triangle case discussed here the values for α are -2.57143, -2.58345 and -2.58661 for the materials with rigid triangles represented by one, two and three nontrivial terms in eqn (9), respectively, when $\eta = 1$. Thus the effect of corners is very small for the rigid inclusion case. For the case of holes it is somewhat larger and $\alpha = 4.14286$, 4.18973 and 4.20187 for the triangular hole given by one, two and three nontrivial terms in eqn (9), respectively, for all admissible η. Note that for the rigid triangle case where corners become sharper, the effective modulus E^c increases, while for the case of cavities the situation is opposite; this tendency is consistent with the observations pertaining to Fig. 1.

The expressions for the other elastic constants, the effective area bulk modulus K^c and the shear modulus G^c are given by

$$\frac{K}{K^c} = 1 + \frac{\eta+1}{\eta-1}\tilde{c}f \tag{39}$$

$$\frac{G}{G^c} = 1 + (\eta+1)\tilde{d}f. \tag{40}$$

Equations (33), (39) and (40) represent the dilute effective elastic constants of materials with rigid polygonal inclusions if the constants $\tilde{c}$ and $\tilde{d}$ defined in the Appendix are used. However, we can also obtain from these results the effective elastic moduli of materials

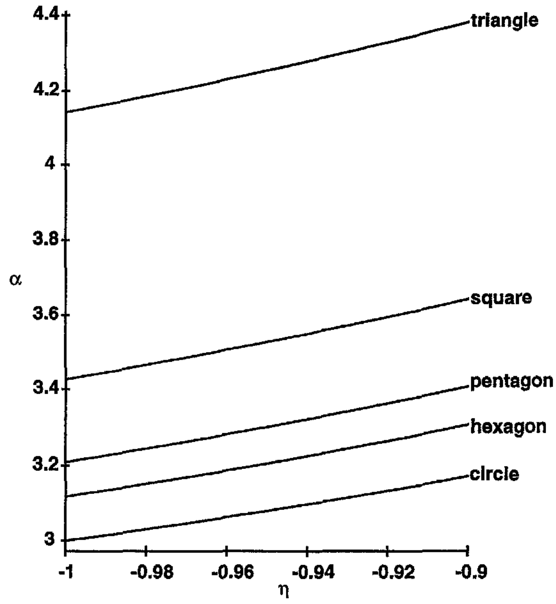

Fig. 3. The parameter α versus η plotted in the unphysical range; the results for materials with cavities of several polygonal shapes approximated by one nontrivial term in eqn (9) are at $\eta = -1$.

with cavities by the use of *Dundurs correspondence*. Dundurs (1989) observed that in plane elasticity if the stresses in an elastic body containing rigid inclusions are known, then by setting $\eta = -1$, we can obtain stresses for the corresponding problems involving cavities. These cavities are in general traction-free but they may have constant shear tractions arising from rotations (Markenscoff, 1993). In the boundary value problems considered in this paper there are no rotations of inclusions due to symmetry and thus the Dundurs correspondence yields solutions for holes with traction-free surfaces. In our problem dealing with the effective planar elastic constants of materials we can also apply this correspondence by setting $\eta = -1$ in the constants $\tilde{c}$ and $\tilde{d}$, while the ηs introduced from the elastic energy remain unchanged. We illustrate it on the example of the effective area bulk modulus K^c. Thus, eqn (39) for rigid inclusions is

$$\frac{K}{K^c} = 1 + \frac{\eta+1}{\eta-1}\,\tilde{c}(\eta)f \tag{41}$$

and for the materials with cavities,

$$\frac{K}{K^c} = 1 + \frac{\eta+1}{\eta-1}\,\tilde{c}(-1)f. \tag{42}$$

The other effective elastic constants, given by eqns (33) and (40), can be written similarly.

This provides us with a unified formulation for the two problems involving the effective elastic moduli of materials with rigid inclusions and cavities; it may also serve as a check of the results on rigid inclusions if the solution for the holes is known. For an easy reference and comparison we give in the Appendix in Table A1 the constants $\tilde{c}$ and $\tilde{d}$ for both rigid inclusions and holes represented by the conformal transformation (9) with one nontrivial term.

The numerical results which involve the Dundurs correspondence concept are given in Figs 3 and 4. These figures correspond to Figs 1 and 2, respectively, but now α is plotted in the unphysical range. The values of α at $\eta = -1$ represent those for materials with holes and they agree with the results given in Jasiuk *et al.* (1994). In the graph we plotted α vs η in the range $-1 \leqslant \eta \leqslant -0.9$ to illustrate the nature of the limit which is well behaved from both the left and right hand sides. We were unable to plot the results for the whole range of intermediate values of η between $-1 < \eta < 1$ as the results became singular in this

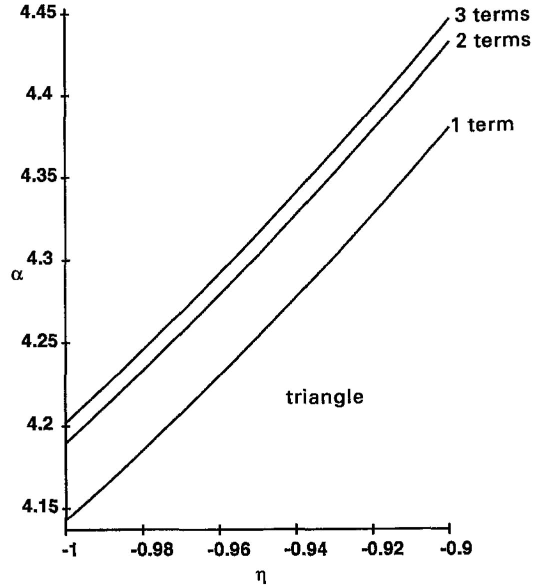

Fig. 4. The parameter α versus η plotted in the unphysical range; the results for materials with triangular cavities approximated by one, two and three nontrivial terms in eqn (9) are at $\eta = -1$.

interval. These singular points can be easily obtained by setting the denominators of $\tilde{c}$ and $\tilde{d}$ to zero. For the shapes considered in this paper and given in the Appendix they occur in the vicinity of $\eta = 0$ and when we use more terms in the transformation (9) for a given polygon the number of singular points increases. For example:

(1) for a triangle approximated by one nontrivial term in the series (9) [case 1(i) in the Appendix] there is one singular point exactly at $\eta = 0$, when two nontrivial terms are used [case 1(ii)] there are two singular points in $\tilde{c}$ at $\eta = 0$ and $\eta = -2/45 = -0.0444$ and two for $\tilde{d}$ at $\eta = \pm 0.0385$, and when three terms are used [case 1(iii)] there are three singular points in $\tilde{c}$ at $\eta = 0$, 0.0064 and -0.0591 and in $\tilde{d}$ at $\eta = \pm 0.0522$ and ± 0.0062;

(2) for the pentagon given by one nontrivial term [case 3(i)] the singular points are at $\eta = 0$ and ± 0.1414 and for two terms at $\eta = 0$ and -0.0533 for $\tilde{c}$ and at $\eta = \pm 0.0102$ and ± 0.1596 for $\tilde{d}$;

(3) for the hexagon approximated by one nontrivial term [case 4(i)] the singular points are at $\eta = 0$ and ± 0.1155, and for two terms [case 4(ii)] at $\eta = 0$ and -0.0505 for $\tilde{c}$ and $\eta = \pm 0.0105$ and ± 0.1349 for $\tilde{d}$.

Thus there is clearly a pattern here that all these points line in the small interval around $\eta = 0$. This may be expected in view of the mathematical theory of elasticity and the Cosserat spectrum (Cosserat, 1898; Mikhlin, 1970; Markenscoff and Paukshto, 1994).

5. EFFECTIVE MEDIUM THEORIES PREDICTIONS

In this section we employ two effective medium theories: the self-consistent method (Budiansky, 1965; Hill, 1965) and the differential scheme (Salganik, 1975; McLaughlin, 1977) to predict the effective elastic moduli of materials with polygonal rigid inclusions and holes. This allows us to account for the inclusions' interactions. We choose these two methods for simplicity. A more accurate study of the effects of inclusions' interactions and specific geometric arrangements could be done, for example, by conducting numerical simulations [e.g. Day *et al.* (1992)].

The self-consistent method assumes that a single inclusion is embedded in an infinite material having the effective properties of a composite. We obtain the governing equation for the self-consistent method by following a simple approach illustrated in Jasiuk *et al.* (1992a,b). We use the dilute results (39) and (40) obtained in section 4 and we achieve self-consistency by replacing the quantities on the right hand sides involving η by η^c.

$$\frac{K}{K^c} = 1 + \frac{\eta^c+1}{\eta^c-1}\,\tilde{c}^c f \tag{43}$$

$$\frac{G}{G^c} = 1 + (\eta^c+1)\,\tilde{d}^c f \tag{44}$$

where $\tilde{c}^c = \tilde{c}(\eta^c)$ and $\tilde{d}^c = \tilde{d}(\eta^c)$.

These equations can be solved by first forming a single equation for η by using, for example, eqn (6):

$$\frac{G}{K} = \tfrac{1}{2}(\eta - 1). \tag{45}$$

Then, if we substitute eqns (43) and (44) into eqn (45), we have

$$\eta - 1 = (\eta^c - 1)\frac{1 + (\eta^c+1)\tilde{d}^c f}{1 + \left(\dfrac{\eta^c+1}{\eta^c-1}\right)\tilde{c}^c f}, \tag{46}$$

which can be solved for η^c as a function of area fraction f and η. After η^c is known the effective moduli K^c and G^c can be easily calculated and E^c can be found from eqn (5). This approach holds for both rigid inclusions and holes with a difference that for the holes $\tilde{c}^c$ and $\tilde{d}^c$ are constant (not functions of η^c) and thus the algebra is simplified.

The differential scheme implies that the composite has a wide distribution of fiber sizes and it involves an iterative procedure in which at every step of iteration larger fibers are placed in a material containing a dilute concentration of smaller fibers. This is repeated many times and in a limit the differential equation is written for each property. This method uses a dilute result as basis and thus eqns (39) and (40) become

$$\frac{\mathrm{d}K^c}{K^c} = -\frac{\mathrm{d}f}{1-f}\frac{\eta^c+1}{\eta^c-1}\tilde{c}^c \tag{47}$$

$$\frac{\mathrm{d}G^c}{G^c} = -\frac{\mathrm{d}f}{1-f}(\eta^c+1)\tilde{d}^c. \tag{48}$$

These are two highly coupled differential equations. The solution can be obtained by using conditions $K^c = K$ and $G^c = G$ at $f = 0$; for more details see, for example, Jasiuk *et al.* (1992a,b).

An interesting quantity to explore is the area fraction of inclusions at percolation, i.e. the area fraction at which the effective elastic moduli go either to infinity (for rigid inclusions) or vanish (for holes). We will refer to this area fraction as f^p. The differential scheme predicts that $f^p = 1$ for all shapes, due to the gradation of sizes assumption. The self-consistent method gives more interesting results. Recall that for a material with circular rigid inclusions it gives a prediction that $f^p = 2/3$, which agrees very well with the percolation concentration for randomly centered and overlapping circular inclusions. We can find the f^p prediction from the self-consistent method by setting the right hand sides of eqns (43) and (44) to zero. Then we have two equations for the two unknowns f^p and $\eta^p = \eta(f^p)$:

$$1 \pm \frac{\eta^p+1}{\eta^p-1}\,\tilde{c}^p f^p = 0 \tag{49}$$

$$1 \pm (\eta^p+1)\tilde{d}^p f^p = 0, \tag{50}$$

where $\tilde{c}^{\mathrm{p}} = \tilde{c}(\eta^{\mathrm{p}})$ and $\tilde{d}^{\mathrm{p}} = \tilde{d}(\eta^{\mathrm{p}})$. The first set of these equations, with $+$, is for a rigid case, and the second, with $-$, is for a hole case. Then eliminating f^{p} from eqns (49) and (50), we find a single equation for η^{p}:

$$c^{\mathrm{p}} - (\eta^{\mathrm{p}} - 1)d^{\mathrm{p}} = 0. \tag{51}$$

For the case of rigid inclusions this equation may have several roots. If we stay in the physical range of η and f and consider only the smallest real root we find $f^{\mathrm{p}} = 0.536$ for triangle, $f^{\mathrm{p}} = 0.642$ for pentagon and $f^{\mathrm{p}} = 0.653$ for hexagon, when one nontrivial term in (9) is used. This indicates that the triangles will percolate first, which is a correct tendency. The corresponding values for η^{p} are $\eta^{\mathrm{p}} = 2.222$, 2.050, 2.029 respectively for these three cases. Recall that for a rigid circle $f^{\mathrm{p}} = 2/3$ when $\eta^{\mathrm{p}} = 2$.

The value of η^{p} is also of interest to us as this is the value toward which the effective η^{c} flows regardless of what the initial Poisson's ratio of the matrix is. We have called it a fixed point (Jasiuk *et al.*, 1992a,b; Day *et al.*, 1992; Jun and Jasiuk, 1993). In Jasiuk *et al.* (1992b) we have studied the fixed point, as predicted by the self-consistent method, for materials with polygonal holes and observed that the values of η^{p} gradually decrease and those of f^{p} gradually increase as the number of sides in the polygon increases until they reach the values $\eta^{\mathrm{p}} = 2$ at $f^{\mathrm{p}} = 1/3$ for the circular shape. We can easily obtain these results here by using Table A1. Equation (51) gives a single value η^{p} for the case of cavities such that $\eta^{\mathrm{p}} = \tilde{c}^{\mathrm{p}}/\tilde{d}^{\mathrm{p}} + 1$ and the corresponding area fraction at percolation $f^{\mathrm{p}} = 1/[\tilde{d}^{\mathrm{p}}(\eta^{\mathrm{p}} + 1)]$. Then, for the cavity shapes given by one nontrivial term in (g) the results are as follows:

(1) for the triangle $\eta^{\mathrm{p}} = 20/9 = 2.222$ and $f^{\mathrm{p}} = 7/29 = 0.241$;
(2) for the pentagon $\eta^{\mathrm{p}} = 2.019$ and $f^{\mathrm{p}} = 0.312$,
(3) for the hexagon $\eta^{\mathrm{p}} = 2.009$ and $f^{\mathrm{p}} = 0.321$.

Again these results show that the triangles will percolate first.

It may be pointed out here that the existence of fixed point was proved rigorously for the materials with holes by Day *et al.* (1992) and Thorpe and Jasiuk (1992) by using the CLM theorem (Cherkaev *et al.*, 1992). Normally for the rigid inclusion case there is no fixed point, except in special cases (Chen *et al.*, 1994). However, there is a tendency of the effective Poisson's ratio to flow towards this point. Thus, the predictions of the self-consistent method presented here are not accurate but they seem to obey the correct trends. For a more detailed study of the material response at percolation for arbitrary shapes see e.g. Garboczi *et al.* (1991).

Note also that the Dundurs correspondence (Dundurs, 1989) was successfully used in this paper for the prediction of effective elastic moduli of materials with regular polygon shapes. However, it needs to be used with caution for more irregular shapes as the rotations may arise (Markenscoff, 1993). Also, at higher volume fractions, when there is inclusions' interaction, the rigid inclusions may rotate. However, in the two approximate methods: self-consistent and differential schemes, used in this paper, the rotations due to interactions do not enter since these two methods are based on single inclusion solutions and dilute results.

6. CONCLUSIONS

In this paper we predicted the effective elastic constants of composite materials containing the polygonal rigid inclusions and cavities by using the Dundurs correspondence between the rigid inclusions and cavities. In the numerical results we studied the effects of geometry and Poisson's ratio on the effective elastic properties and pointed out the differences in physical responses of these two types of composite materials. We found that the effect of sharp corners, which gave rise to high local stresses, on the effective elastic constants is small and it is smaller for the case of rigid inclusions. The rigid equilateral triangle shape gave the higher elastic modulus in the dilute limit than the other regular polygons, while the opposite trend was observed in the case of holes, where the circular shape gave the optimum elastic response. We have also studied the material response at high area fractions by using the self-consistent method and explored the material behavior at percolation.

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APPENDIX

In this Appendix we give the expressions for the constants $\tilde{c}$ and $\tilde{d}$ for the rigid polygonal inclusions approximated by one, two and three nontrivial terms in transformation (9).

1. *Triangle*

(i)

$$w(\zeta) = R\left(\zeta + \frac{1}{3\zeta^2}\right)$$

$$\tilde{c} = -\frac{(9\eta^2 - 11\eta + 2)}{14\eta}, \quad \tilde{d} = -\frac{9}{7\eta}.$$

(ii)

$$w(\zeta) = R\left(\zeta + \frac{1}{3\zeta^2} + \frac{1}{45\zeta^5}\right)$$

$$\tilde{c} = -\frac{(18225\eta^3 - 21690\eta^2 + 3463\eta + 2)}{628(45\eta + 2)\eta}, \quad \tilde{d} = b - \frac{273375\eta}{314(675\eta^2 - 1)}.$$

(iii)

$$w(\zeta) = R\left(\zeta + \frac{1}{3\zeta^2} + \frac{1}{45\zeta^5} + \frac{1}{162\zeta^8}\right)$$

$$\tilde{c} = -\frac{(1076168025\eta^4 - 1275195960\eta^3 + 199536912\eta^2 - 508852\eta - 125)}{25424(65610\eta^2 + 3456\eta - 25)\eta}$$

$$\tilde{d} = -\frac{1793613375\eta(2187\eta^2 - 1)}{12712(239148450\eta^4 - 661689\eta^2 + 25)}.$$

2. *Square*

(i)

$$w(\zeta) = R\left(\zeta + \frac{1}{6\zeta^3}\right)$$

$$\tilde{c} = -\frac{(12\eta^2 - 13\eta + 1)}{22\eta}, \quad \tilde{d} = -\frac{72}{11(6\eta - 1)}.$$

(ii)

$$w(\zeta) = R\left(\zeta + \frac{1}{6\zeta^3} + \frac{1}{56\zeta^7}\right)$$

$$\tilde{c} = -\frac{(75264\eta^3 - 78008\eta^2 + 2735\eta + 9)}{2458\eta(56\eta + 3)}, \quad \tilde{d} = \frac{12644352\eta}{1229(-9408\eta^2 + 1637\eta + 15)}$$

(iii)

$$w(\zeta) = R\left(\zeta + \frac{1}{6\zeta^3} + \frac{1}{56\zeta^7} + \frac{1}{176\zeta^{11}}\right)$$

$$\tilde{c} = -\frac{(12822577152\eta^4 - 13203550976\eta^3 + 380071604\eta^2 + 905307\eta - 3087)}{108110\eta(216832\eta^2 + 13464\eta - 147)}$$

$$\tilde{d} = -\frac{269274120192\eta(176\eta + 5)}{54055(801411072\eta^3 + 165198880\eta^2 + 2096842\eta - 6615)}.$$

Square (rotated 45°)

(i)

$$w(\zeta) = R\left(\zeta - \frac{1}{6\zeta^3}\right)$$

$$\tilde{c} = -\frac{(12\eta^2 - 13\eta + 1)}{22\eta}, \quad \tilde{d} = -\frac{72}{11(6\eta + 1)}.$$

(ii)

$$w(\zeta) = R\left(\zeta - \frac{1}{6\zeta^3} + \frac{1}{56\zeta^7}\right)$$

$$\tilde{c} = -\frac{(75264\eta^3 - 78008\eta^2 + 2735\eta + 9)}{2458\eta(56\eta + 3)}, \quad \tilde{d} = \frac{12644352\eta}{1229(-9408\eta^2 - 1637\eta + 15)}.$$

(iii)

$$w(\zeta) = R\left(\zeta - \frac{1}{6\zeta^3} + \frac{1}{56\zeta^7} - \frac{1}{176\zeta^{11}}\right)$$

$$\tilde{c} = -\frac{(12822577152\eta^4 - 13203550976\eta^3 + 380071604\eta^2 + 905307\eta - 3087)}{108110\eta(216832\eta^2 + 13464\eta - 147)}$$

$$\tilde{d} = -\frac{269274120192\eta(176\eta + 5)}{54055(801411072\eta^3 - 165198880\eta^2 + 2096842\eta + 6615)}.$$

3. *Pentagon*

(i)

$$w(\zeta) = R\left(\zeta + \frac{1}{10\zeta^4}\right)$$

$$\tilde{c} = -\frac{(25\eta^2 - 26\eta + 1)}{48\eta}, \quad \tilde{d} = -\frac{625\eta}{12(50\eta^2 - 1)}.$$

(ii)

$$w(\zeta) = R\left(\zeta + \frac{1}{10\zeta^4} + \frac{1}{75\zeta^9}\right)$$

$$\tilde{c} = -\frac{(46875\eta^3 - 464325\eta^2 - 454\eta + 4)}{1198(75\eta + 4)\eta}$$

$$\tilde{d} = -\frac{7031250\eta(1875\eta^2 - 4)}{599(21093750\eta^4 - 539325\eta^2 + 56)}.$$

(ii)

$$w(\zeta) = R\left(\zeta + \frac{1}{10\zeta^4} + \frac{1}{75\zeta^9} + \frac{4}{875\zeta^{14}}\right)$$

$$\tilde{c} = -\frac{(83740234375\eta^4 - 82409578125\eta^3 - 1363897150\eta^2 + 33259332\eta - 18432)}{69862\eta(2296875\eta^2 + 139300\eta - 1728)}$$

$$\tilde{d} = -\frac{502441406250\eta(251220703125\eta^4 - 1029017500\eta^2 + 101376)}{34931(3462135314941406250\eta^6 - 99324626551171875\eta^4 + 26796690960\eta^2 - 350355456)}.$$

4. *Hexagon*

(i)

$$w(\zeta) = R\left(\zeta + \frac{1}{15\zeta^5}\right)$$

$$\tilde{c} = -\frac{(45\eta^2 - 46\eta + 1)}{88\eta}, \quad \tilde{d} = -\frac{3375\eta}{44(75\eta^2 - 1)}.$$

(ii)

$$w(\zeta) = R\left(\zeta + \frac{1}{15\zeta^5} + \frac{1}{99\zeta^{11}}\right)$$

$$\tilde{c} = -\frac{(441045\eta^3 - 429561\eta^2 - 11509\eta + 25)}{8702(99\eta + 5)\eta}$$

$$\tilde{d} = -\frac{121287375\eta(3367\eta^2 - 7)}{4351(88944075\eta^4 - 1580986\eta^2 + 175)}.$$

In addition, for an easy reference and a comparison of rigid inclusion and cavity cases we give, in Table A1, the constants for one term transformation

$$w(\zeta) = R\left(\zeta + \frac{2}{n(n-1)\zeta^{n-1}}\right),$$

where n is the number of sides in the polygon.

Table A1. Constants $\tilde{c}$ and $\tilde{d}$ for rigid inclusions and holes (one term approximation)

Shape	$\tilde{c}$ (rigid inclusion)	$\tilde{c}$ (hole, $\eta = -1$)	$\tilde{d}$ (rigid inclusion)	$\tilde{d}$ (hole, $\eta = -1$)
Circle	$-\frac{(\eta-1)}{2}$	1	$-\frac{1}{\eta}$	1
Triangle	$-\frac{(9\eta-2)(\eta-1)}{14\eta}$	$\frac{11}{7}$	$-\frac{9}{7\eta}$	$\frac{9}{7}$
Square	$-\frac{(12\eta-1)(\eta-1)}{22\eta}$	$\frac{13}{11}$	$-\frac{72}{11(6\eta-1)}$	$\frac{72}{77}$
Square (rotated 45°)	$-\frac{(12\eta-1)(\eta-1)}{22\eta}$	$\frac{13}{11}$	$-\frac{72}{11(6\eta+1)}$	$\frac{72}{55}$
Pentagon	$-\frac{(25\eta-1)(\eta-1)}{48\eta}$	$\frac{13}{12}$	$-\frac{625\eta}{12(50\eta^2-1)}$	$\frac{625}{588}$
Hexagon	$-\frac{(45\eta-1)(\eta-1)}{88\eta}$	$\frac{23}{22}$	$-\frac{3375\eta}{44(75\eta^2-1)}$	$\frac{3375}{3256}$